THE HOMOTOPY TYPE OF THE COMPLEMENT OF THE CODIMENSION-TWO COORDINATE SUBSPACE ARRANGEMENT

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A complex coordinate subspace of $\mathbb{C}^n$ is given by

$$L_\sigma = \{(z_1, \dots, z_n) \in \mathbb{C}^n \,|\, z_{i_1} = \cdots = z_{i_k} = 0\}$$

where $\sigma = \{i_1, \dots, i_k\}$ is a subset of $[m]$. For each simplicial complex K on the set $[m]$ we associate the complex coordinate subspace arrangement $\mathcal{CA}(K) = \{L_\sigma \,|\, \sigma \notin K\}$ and its complement $U(K) = \mathbb{C}^n \setminus \bigcup_{\sigma \notin K} L_\sigma$. On the other hand, to K we can associate the Davis -Januszkiewicz space $DJ(K) = \bigcup_{\sigma\in K} BT_\sigma \subset BT^n$, where BT^n is the classifying space of n-dimensional torus, that is, the product of n copies of infinite-dimensional projective space $\mathbb{C}P^\infty$, and $BT_\sigma := \{(x_1, \dots, x_n) \in BT^n | x_i = *$ where $i \notin \sigma\}$. Let $\mathcal{Z}_K$ be the fibre of $DJ(K) \longrightarrow BT^n$. By [BP, 8.9], there is an equivariant deformation retraction $U(K) \longrightarrow \mathcal{Z}_K$, and the integral cohomology of $\mathcal{Z}_K$ has been calculated in [BP, 7.6 and 7.7].

Theorem 1. *The complement of the codimension-two coordinate subspace arrangement in $\mathbb{C}^n$ has the homotopy type of the wedge of spheres*

$$\bigvee_{k=2}^{n} (k-1)\binom{n}{k} S^{k+1}.$$

Proof. Let K be a disjoint union of n vertices. Then $DJ(K)$ is the wedge of n copies of $\mathbb{C}P^\infty$ and $U(K)$ is the complement of the set of all codimension-two coordinates subspaces $z_i = z_j = 0$ for $1 \le i < j \le n$ in $\mathbb{C}^n$. Therefore to prove the theorem we have to determine the homotopy fibre of the inclusion $\bigvee_{t=1}^n \mathbb{C}P^\infty \longrightarrow \prod_{t=1}^n \mathbb{C}P^\infty$. This is done by applying Proposition 5 to the case $X_1 = \cdots = X_n = \mathbb{C}P^\infty$ and noting that $\Omega\mathbb{C}P^\infty \simeq S^1$. $\square$

It should be emphasized that Theorem 1 holds without suspending. Previously, decompositions were known only after some number of suspensions, the best of which was by Schaper [S] who required one suspension. To finish the proof of Theorem 1 it remains to prove Proposition 5. This was originally proved by Porter [P] by examining subspaces of contractible spaces. We present an accelerated proof based on the Cube Lemma.

We work in the category of based, connected topological spaces and continuous maps. Let $*$ denote the basepoint. For spaces X, Y, let $X \rtimes Y = (X\times Y)/(*\times Y)$, $X \wedge Y = (X \rtimes Y)/(X \times *)$, and $X * Y = \Sigma X \wedge Y$. Denote the identity map on X by X. Denote the map which sends all points to the basepoint by $*$.

Lemma 2. *Let A, B, and C be spaces. Define Q as the homotopy pushout of the map $A \times B \xrightarrow{*\times B} C \times B$ and the projection $A \times B \xrightarrow{\pi_1} A$. Then $Q \simeq (A * B) \vee (C \rtimes B)$.*

Proof. Consider the diagram of iterated homotopy pushouts

$$\begin{tikzcd}
A \times B \arrow[r, "\pi_2"] \arrow[d, "\pi_1"] & B \arrow[r, "i_2"] \arrow[d, "*"] & C \times B \arrow[d, "s"] \\
A \arrow[r, "*"] & A * B \arrow[r, "t"] & \overline{Q}
\end{tikzcd}$$

where π_2, i_2 are the projection and inclusion respectively. Here, it is well known that the left square is a homotopy pushout, and the right homotopy pushout defines $\overline{Q}$. Note that $i_2 \circ \pi_2 \simeq * \times B$. The outer rectangle in an iterated homotopy pushout diagram is itself a homotopy pushout, so $\overline{Q} \simeq Q$. The right pushout then shows that the homotopy cofibre of $C \times B \longrightarrow Q$ is $\Sigma B \vee (A * B)$. Thus t has a left homotopy inverse. Further, $s \circ i_2 \simeq *$ so pinching out B in the right pushout gives a homotopy cofibration $C \rtimes B \longrightarrow Q \xrightarrow{r} A * B$ with $r \circ t$ homotopic to the identity map. $\square$

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Lemma 3. *Let $Y_1, \ldots, Y_n$ be spaces. Then there is a homotopy equivalence*

$$\Sigma(Y_1 \times \cdots \times Y_n) \simeq \bigvee_{k=1}^{n} \left(\bigvee_{1 \leq i_1 < \cdots < i_k \leq n} \Sigma Y_{i_1} \wedge \cdots \wedge Y_{i_k} \right).$$

Proof. Induct on the decomposition $\Sigma(A \times B) = \Sigma A \vee \Sigma B \vee (\Sigma A \wedge B)$. □

The following was proved by Mather [M] and is known as the Cube Lemma.

Lemma 4. *Suppose there is a diagram of spaces and maps*

$$\begin{tikzcd}
E \arrow[rr] \arrow[dr] \arrow[dd] & & F \arrow[dr] \arrow[dd] & \\
& G \arrow[rr] \arrow[dd] & & H \arrow[dd] \\
A \arrow[rr] \arrow[dr] & & B \arrow[dr] & \\
& C \arrow[rr] & & D
\end{tikzcd}$$

where the bottom face is a homotopy pushout and the four sides are obtained by pulling back with $H \longrightarrow D$. Then the top face is a homotopy pushout. □

Proposition 5. *Let $X_1, \ldots, X_n$ be spaces. Consider the homotopy fibration*

$$F_n \longrightarrow X_1 \vee \cdots \vee X_n \longrightarrow X_1 \times \cdots \times X_n$$

obtained by including the wedge into the product. Then there is a homotopy decomposition

$$F_n \simeq \bigvee_{k=2}^{n} \left(\bigvee_{1 \leq i_1 < \cdots < i_k \leq n} (k-1)(\Sigma\Omega X_{i_1} \wedge \cdots \wedge \Omega X_{i_k}) \right).$$

Proof. We induct on n. When $n = 2$ it is well known that $F_2 \simeq \Sigma\Omega X_1 \wedge \Omega X_2$. Let $n \geq 3$ and assume the Proposition holds for F_{n-1}. Let $M_k = X_1 \vee \cdots \vee X_k$ and $N_k = X_1 \times \cdots \times X_k$. Observe that M_n is the pushout of M_{n-1} and X_n over a point. Composing each vertex of the pushout into N_n we obtain homotopy fibrations $\Omega N_n \longrightarrow * \longrightarrow N_n$, $\Omega N_{n-1} \longrightarrow X_n \longrightarrow N_n$, $F_{n-1} \times \Omega X_n \longrightarrow M_{n-1} \longrightarrow N_n$, and $F_n \longrightarrow M_n \longrightarrow N_n$. Write N_n as $N_{n-1} \times X_n$. Then Lemma 4 implies that there is a homotopy pushout

$$\begin{tikzcd}
\Omega N_{n-1} \times \Omega X_n \arrow[r, "h"] \arrow[d, "g"] & F_{n-1} \times \Omega X_n \arrow[d] \\
\Omega N_{n-1} \arrow[r] & F_n
\end{tikzcd}$$

where g is easily identified as the projection and h is the connecting map for the homotopy fibration $F_{n-1} \times \Omega X_n \longrightarrow M_{n-1} \times * \longrightarrow N_{n-1} \times X_n$. So $h \simeq \partial_{n-1} \times \Omega X_n$ where ∂_{n-1} is the connecting map of the fibration $F_{n-1} \longrightarrow M_{n-1} \longrightarrow N_{n-1}$. But $\partial_{n-1} \simeq *$ as $\Omega M_{n-1} \longrightarrow \Omega N_{n-1}$ has a right homotopy inverse. Thus $h \simeq * \times \Omega X_n$. By Lemma 2, $F_n \simeq (\Omega N_{n-1} * \Omega X_n) \vee (F_{n-1} \rtimes \Omega X_n)$. Since F_{n-1} is a suspension, $F_{n-1} \rtimes \Omega X_n \simeq F_{n-1} \vee (F_{n-1} \wedge \Omega X_n)$. Combining the decomposition of $\Sigma\Omega N_n \simeq \Sigma(\Omega X_1 \times \cdots \times \Omega X_n)$ in Lemma 3 with the inductive decomposition of F_{n-1} and collecting like terms, the asserted wedge decomposition of F_n follows. □

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